Ballistic transport in random magnetic fields with anisotropic long-ranged correlations

2005 J. Phys. A: Math. Gen. 38 L235

(http://iopscience.iop.org/0305-4470/38/14/L05)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) L235–L240 doi:10.1088/0305-4470/38/14/L05

LETTER TO THE EDITOR

Ballistic transport in random magnetic fields with anisotropic long-ranged correlations

Hajo Leschke[1], **Simone Warzel**[2] **and Alexandra Weichlein**[1]

[1] Institut für Theoretische Physik, Universität Erlangen-Nürnberg, Staudtstrasse 7, 91058 Erlangen, Germany
[2] Jadwin Hall, Princeton University, Princeton, NJ 08544, USA

E-mail: swarzel@princeton.edu

Received 8 November 2004, in final form 13 January 2005
Published 21 March 2005
Online at stacks.iop.org/JPhysA/38/L235

Abstract
We present exact theoretical results about energetic and dynamic properties of a spinless charged quantum particle on the Euclidean plane subjected to a perpendicular random magnetic field of Gaussian type with non-zero mean. Our results refer to the simplifying but remarkably illuminating limiting case of an infinite correlation length along one direction and a finite but strictly positive correlation length along the perpendicular direction in the plane. They are therefore 'random analogues' of results first obtained by Iwatsuka in 1985 and by Müller in 1992, which are greatly esteemed, in particular for providing a basic understanding of transport properties in certain quasi-two-dimensional semiconductor heterostructures subjected to non-random inhomogeneous magnetic fields.

PACS numbers: 72.15.Gd, 72.20.My, 73.23.Ad, 75.47.Jn

Quantum-mechanical models for a single spinless electrically charged particle on the (infinitely extended) Euclidean plane $\mathbb{R}^2$ subjected to a perpendicular spatially random magnetic field (RMF) have become a topic of growing interest over the last decade. Such models are currently discussed in relation with magneto-transport properties of quasi-two-dimensional semiconductor heterostructures with certain randomly built-in magnets [1–7]. Moreover, they are part of effective theories for the fractional quantum Hall effect [8–10]. Just as in Anderson's model [11] of a quantum particle in a random scalar potential (and no or a constant magnetic field)[3], the fundamental question is to understand the spectral and transport properties of the underlying Hilbert-space operator representing the (kinetic) energy and generating the dynamics of the particle in a RMF. Until recently, different studies by perturbative, quasi-classical, field-theoretical and numerical methods have given partially conflicting answers [13–28].

[3] For a recent survey of rigorous results in the case of continuum models see [12].

Since 'the power and utility of simple models can hardly be overestimated' [30], the purpose of this letter is to present first exact (de)localization results in case of simple, but remarkably illuminating RMFs[4]. The simplification arises from the assumption that the fluctuations of the RMF on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ are anisotropically long-ranged correlated in the sense that we consider the limiting case of an infinite correlation length along one direction and take the correlation length to be finite but strictly positive along the perpendicular direction in the plane. In other words, we assume the RMF to be independent of one of the two Cartesian co-ordinates, which we choose to be the second one, x_2. The remaining dependence of the RMF values on the first co-ordinate x_1 we suppose to be governed by the realizations $b := \{b(x_1)\}_{x_1 \in \mathbb{R}}$ of a homogeneous and *ergodic* real-valued random (or stochastic) process with the real line $\mathbb{R} =]-\infty, \infty[$ as its parameter set [32]. We will assume throughout that its mean $\overline{b(0)}$ is non-zero and finite,

$$0 < \left|\, \overline{b(0)} \,\right| < \infty. \tag{1}$$

Here the overbar denotes the probabilistic (or ensemble) average. Taking the (Lebesgue-) integral $a_2(x_1) := \int_0^{x_1} \mathrm{d}x_1' b(x_1')$, which exists almost surely for all $x_1 \in \mathbb{R}$, as the second component of the vector potential $(0, a_2(x_1))$ in the asymmetric gauge, the Hamiltonian (or kinetic-energy operator) is then given as

$$H := \tfrac{1}{2}\big[P_1^2 + (P_2 - a_2(Q_1))^2\big] \tag{2}$$

in terms of the two components of the usual canonical momentum and position operators, P_1, P_2, respectively $Q_1, (Q_2,)$ corresponding to the x_1- and the x_2-direction. All operators act self-adjointly on the Hilbert space $\mathrm{L}^2(\mathbb{R}^2) = \mathrm{L}^2(\mathbb{R}) \otimes \mathrm{L}^2(\mathbb{R})$ of square-integrable, complex-valued functions on the plane $\mathbb{R}^2$. For notational transparency we use physical units such that Planck's constant (divided by 2π) and the particle's mass and charge are all equal to 1.

Energetic properties. The nice feature of H is its translational invariance along the x_2-direction so that it commutes with P_2, an operator which can be partially Fourier decomposed on $\mathrm{L}^2(\mathbb{R}^2)$ according to $P_2 = \int_{-\infty}^{\infty} \mathrm{d}k\, k \mathbb{1} \otimes |k\rangle\langle k|$ (using an informal notation). Therefore, H can be decomposed according to

$$H = \int_{-\infty}^{\infty} \mathrm{d}k\, H(k) \otimes |k\rangle\langle k| \tag{3}$$

into the one-parameter family

$$H(k) := \tfrac{1}{2}\big[P_1^2 + (k\mathbb{1} - a_2(Q_1))^2\big], \qquad k \in \mathbb{R}, \tag{4}$$

of effective (or fibre) Hamiltonians on the Hilbert space $\mathrm{L}^2(\mathbb{R})$ for the one-dimensional motion along the x_1-direction. Here each wave number $k \in \mathbb{R}$ is a possible (spectral) value of the particle's canonical momentum along the x_2-direction. For any typical realization b the Birkhoff–Khinchin ergodic theorem [32, 33],

$$\lim_{|x_1| \to \infty} \frac{a_2(x_1)}{x_1} = \overline{b(0)} \neq 0, \tag{5}$$

ensures that the effective scalar potential entering $H(k)$ confines the particle along the x_1-direction for large $|x_1|$ quadratically. As a consequence, for each fixed $k \in \mathbb{R}$ the operator $H(k)$ has purely discrete spectrum with strictly positive and non-degenerate eigenvalues $0 < \varepsilon_0(k) < \varepsilon_1(k) < \ldots$ so that its spectral resolution reads

$$H(k) = \sum_{n=0}^{\infty} \varepsilon_n(k) |\varphi_n(k)\rangle\langle\varphi_n(k)| \tag{6}$$

[4] Reference [31] outlines a rigorous proof of the existence of localized states at low energies for certain RMFs on the (infinite) square lattice $\mathbb{Z}^2$ instead of the two-dimensional continuum $\mathbb{R}^2$.

with normalized and pairwise orthogonal eigenfunctions $|\varphi_0(k)\rangle$, $|\varphi_1(k)\rangle$, ... spanning $\mathrm{L}^2(\mathbb{R})$. By (3) and (6) the spectrum of H is given by a set-theoretic union,

$$\operatorname{spec} H = \bigcup_{n=0}^{\infty} \beta_n, \qquad \beta_n := \Big[\inf_{k\in\mathbb{R}} \varepsilon_n(k), \sup_{k\in\mathbb{R}} \varepsilon_n(k)\Big]. \tag{7}$$

Here the closed interval β_n is the nth energy band. It is a subset of the positive half-line $[0,\infty[$ and extends from the lower to the upper edge of the nth energy-band function ε_n. A further important consequence of the assumed ergodicity is that, although the spectrum of $H(k)$ for fixed $k \in \mathbb{R}$ in general depends on b, each resulting energy band β_n of H is non-random almost surely, that is, the same for all typical b.

The random Hamiltonian H with the non-random energy-band structure of its spectrum is a random variant of models first investigated in [34] and (non-rigorously) in the often quoted paper [35]. By studying special non-random b these and other works [36–42] have illustrated that a non-constant b has a tendency to delocalize the particle along the x_2-direction. In fact, according to classical mechanics a particle with non-zero kinetic energy wanders off to infinity along snake or cycloid-like orbits winding around (straight) contours of constant magnetic field [43, 35]. The quantum analogue of this unbounded motion should manifest itself in the exclusive appearance of (absolutely) continuous spectrum of H, or equivalently, of only strictly positive bandwidths, $|\beta_n| := \sup_{k\in\mathbb{R}} \varepsilon_n(k) - \inf_{k\in\mathbb{R}} \varepsilon_n(k) > 0$ for all n. While plausible from the (quasi-)classical picture, the rigorous exclusion of flat energy bands is not trivial and has been accomplished so far only for certain classes of non-constant[5] but non-random b [34, 37]. Our main theorem establishes for the first time such a result in the random case.

Theorem. *If the RMF is given by a homogeneous Gaussian random process with its mean* $\overline{b(0)}$ *obeying (1) and its covariance function*

$$c(x_1) := \overline{b(x_1)b(0)} - (\overline{b(0)})^2 \tag{8}$$

fulfilling the following two requirements:

(i) *c is continuous at the origin (and hence everywhere) with* $0 < c(0) < \infty$,

(ii) $\lim_{\ell\to\infty} \ell^{-1} \int_0^\ell \mathrm{d}x_1 (c(x_1))^2 = 0$,

then $|\beta_n| > 0$ *for all energy-band indices n and* $\operatorname{spec} H = [0,\infty[$, *almost surely.*

Given our simplifying *a priori* assumption, the two requirements are both mathematically mild and physically relevant. By the first one the RMF is neither non-random nor delta-correlated and has realizations which are continuous in the mean-square sense. Because of the Bochner–Khinchin [32, 33], the Fomin–Grenander–Mayurama [32, 33], and the Wiener theorem [33, 43], the second requirement is then equivalent to the ergodicity of the underlying Gaussian random process. In particular, (ii) requires that the correlation of the RMF's fluctuations at two different points in the plane exhibits some decay with increasing absolute difference of their first co-ordinates. The simple condition $\lim_{|x_1|\to\infty} c(x_1) = 0$ is sufficient but not necessary.

The basic observation for the proof of the theorem is that (i) and (ii) ensure a non-zero probability for the occurrence of realizations b with arbitrarily small absolute values on spatial average over arbitrarily long line segments, that is

$$\operatorname{Prob}\left\{\int_{-\ell}^{\ell} \mathrm{d}x_1 |b(x_1| < \delta\right\} > 0 \tag{9}$$

[5] In the constant case, that is, $b(x_1) = b_0$ for all $x_1 \in \mathbb{R}$ with some constant $b_0 \neq 0$, each eigenvalue of $H(k)$ is independent of $k \in \mathbb{R}$ and given by a Landau level [44, 45], $\varepsilon_n(k) = (n + 1/2)|b_0|$, so that $|\beta_n| = 0$ for all n.

for all $\ell > 0$ and all $\delta > 0$. Such realizations, although rare because of $\overline{b(0)} \neq 0$, come with nearly free motion. More precisely, for any (arbitrarily small) energy $\varepsilon > 0$ and any (arbitrarily large) integer $n_0 \geqslant 0$ there occur realizations such that the effective Hamiltonian $H(0)$ has $n_0 + 1$ eigenvalues strictly smaller than ε. Thanks to the non-randomness of each β_n, this rules out a flat energy band of H at ε. Otherwise the number of eigenvalues of $H(0)$ below ε would be uniformly bounded in the randomness. By a similar argument the almost-sure (purely absolutely continuous) spectrum of H is seen to coincide with the entire positive half-line.

Dynamic properties. As suggested by the (quasi-)classical picture, the non-existence of flat energy bands as supplied by the theorem should come with ballistic transport along the x_2-direction. To prepare a precise statement, we temporarily return to a typical realization b of a general ergodic random process obeying (1). Then (3) and (6) imply that any normalized wave packet $|\psi_0\rangle$ in $\mathrm{L}^2(\mathbb{R}^2)$ with almost surely finite (time-invariant) kinetic energy, $\langle\psi_0|H|\psi_0\rangle < \infty$, and (initial) localization along the x_2-direction in the sense that $\langle\psi_0|Q_2^2|\psi_0\rangle < \infty$, has an asymptotic velocity in the sense that the following (strong) long-time-limit relation holds[6]:

$$\lim_{t\to\infty} t^{-1}\, \mathrm{e}^{\mathrm{i}tH} Q_2\, \mathrm{e}^{-\mathrm{i}tH} |\psi_0\rangle = V_{2,\infty}|\psi_0\rangle. \tag{10}$$

Here the (random) asymptotic velocity operator

$$V_{2,\infty} := \int_{-\infty}^{\infty} \mathrm{d}k\, V_{2,\infty}(k) \otimes |k\rangle\langle k| \tag{11}$$

on $\mathrm{L}^2(\mathbb{R}^2)$ is related to the derivatives of the energy-band functions similarly as in the quantum theory of single electrons in perfect crystals (without external fields) [48],

$$V_{2,\infty}(k) := \sum_{n=0}^{\infty} \frac{\mathrm{d}\varepsilon_n(k)}{\mathrm{d}k}\, |\varphi_n(k)\rangle\langle\varphi_n(k)|, \qquad k \in \mathbb{R}. \tag{12}$$

If the energy band β_n is not flat, $|\beta_n| > 0$, the (random) group velocity $\mathrm{d}\varepsilon_n(k)/\mathrm{d}k$ vanishes at most at countably many $k \in \mathbb{R}$, because $\varepsilon_n(k)$ is an analytic function of k, almost surely. Moreover, by the Feynman–Hellmann theorem, the positivity of the quantum-mechanical variance and the strict inequality $\langle\varphi_n(k)|P_1^2|\varphi_n(k)\rangle > 0$, equations (4) and (6) give the upper estimate $(\mathrm{d}\varepsilon_n(k)/\mathrm{d}k)^2 < 2\varepsilon_n(k)$ (cf [37]). Taken together, this proves the

Corollary. *Under the assumptions of the theorem the particle's motion along the x_2-direction is ballistic in the sense that (10) holds with* $0 < \langle\psi_0|V_{2,\infty}^2|\psi_0\rangle < 2\langle\psi_0|H|\psi_0\rangle < \infty$*, almost surely.*

In contrast, the particle's motion along the x_1-direction is bounded. Indeed, for a typical realization b of a general ergodic random process obeying (1) the quadratic confinement of the particle along the x_1-direction for large $|x_1|$ (cf (5)) implies that any normalized wave packet $|\psi_0\rangle$ in $\mathrm{L}^2(\mathbb{R}^2)$ with almost surely finite kinetic energy and (initial) localization along the x_1-direction in the sense that $\langle\psi_0|Q_1^2|\psi_0\rangle < \infty$, remains localized in the course of time,

$$\sup_{t\in\mathbb{R}} \langle\psi_0|\, \mathrm{e}^{\mathrm{i}tH} Q_1^2\, \mathrm{e}^{-\mathrm{i}tH} |\psi_0\rangle < \infty. \tag{13}$$

[6] The rigorous derivation of (10) is based on the integral form of the Heisenberg equation of motion for $\mathrm{e}^{\mathrm{i}tH} Q_2\, \mathrm{e}^{-\mathrm{i}tH}$. It is similar to that of the corresponding statement for motion in a periodic scalar potential in [46]. For details see [47].

Concluding remarks. Bounds on the Lifshits tail, that is, on the low-energy asymptotics of the integrated density of states have been derived in [49] under the assumptions of the theorem (but allowing for $\overline{b(0)} = 0$).

For further details, complete proofs, and non-Gaussian RMFs obeying (1) and (9) and hence yielding almost surely purely continuous energy spectrum and ballistic transport along the x_2-direction, we refer to [47].

Acknowledgments

We are indebted to Ludwig Schweitzer (Braunschweig, Germany) for hints to the literature. This work was partially supported by the Deutsche Forschungsgemeinschaft (DFG) under grant nos Le 330/12 and Wa 1699/1.

References

[1] Geim A K, Bending S J, Grigorieva I V and Blamire M G 1994 *Phys. Rev.* B **49** 5749
[2] Smith A, Taboryski R, Hansen L T, Sørensen C B, Hedegård P and Lindelof P E 1994 *Phys. Rev.* B **50** R14726
[3] Mancoff F B, Clarke R M, Marcus C M, Zhang S C, Campman K and Gossard A C 1995 *Phys. Rev.* B **51** 13269
[4] Ando M, Endo A, Katsumoto S and Iye Y 2000 *Physica* B **284–288** 1900
[5] Rushforth A W, Gallagher B L, Main P C, Neumann A C, Marrows C H, Zoller I, Howson M A, Hickey B J and Henini M 2000 *Physica* E **6** 751
[6] Bykov A A, Gusev G M, Leite J R, Bakarov A K, Goran A V, Kudryashev V M and Toropov A I 2002 *Phys. Rev.* B **65** 035302
[7] Rushforth A W, Gallagher B L, Main P C, Neumann A C, Henini M, Marrows C H and Hickey B J 2004 *Phys. Rev.* B **70** 193313
[8] Heinonen O (ed) 1998 *Composite Fermions* 2nd edn (Singapore: World Scientific)
[9] Wölfle P 2000 *Advances in Solid State Physics* vol 40 ed B Kramer (Braunschweig: Vieweg) p 77
[10] Murthy M and Shankar R 2003 *Rev. Mod. Phys.* **75** 1101
[11] Anderson P W 1958 *Phys. Rev.* **109** 1492
[12] Leschke H, Müller P and Warzel S 2003 *Markov Process. Relat. Fields* **9** 729
[13] Aronov A G, Mirlin A D and Wölfle P 1994 *Phys. Rev.* B **49** 16609
[14] Lee D K K and Chalker J T 1994 *Phys. Rev. Lett.* **72** 1510
[15] Kawarabayashi T and Ohtsuki T 1995 *Phys. Rev.* B **51** 10897
[16] Sheng D N and Weng Z Y 1995 *Phys. Rev. Lett.* **75** 2388
[17] Yang K and Bhatt R N 1996 *Phys. Rev.* B **55** R1922
[18] Batsch M, Schweitzer L and Kramer B 1998 *Physica* B **249–251** 792
[19] Evers F, Mirlin A D, Polyakov D G and Wölfle P 1999 *Phys. Rev.* B **60** 8951
[20] Furusaki A 1999 *Phys. Rev. Lett.* **82** 604
[21] Potempa H and Schweitzer L 1999 *Ann. Phys. (Lpz)* **8** SI 209
[22] Kawarabayashi T, Kramer B and Ohtsuki T 1999 *Ann. Phys. (Lpz)* **8** 487
[23] Sheng D N and Weng Z Y 2000 *Europhys. Lett.* **50** 776
[24] Yakubo K 2000 *Phys. Rev.* B **62** 16756
[25] Taras-Semchuk D and Efetov K B 2000 *Phys. Rev. Lett.* **85** 1060
Mirlin A D and Wölfle P 2001 *Phys. Rev. Lett.* **86** 3688 (comment)
Taras-Semchuk D and Efetov K B 2001 *Phys. Rev. Lett.* **86** 3689 (reply)
[26] Taras-Semchuk D and Efetov K B 2001 *Phys. Rev.* B **64** 115301
[27] Nguyen H K 2002 *Phys. Rev.* B **66** 144201
[28] Kawarabayashi T and Ohtsuki T 2003 *Phys. Rev.* B **67** 165309
[29] Efetov K B and Kogan V R 2003 *Phys. Rev.* B **68** 245313
[30] Kittel C 1966 *Introduction to Solid State Physics* 3rd edn (New York: Wiley) preface
[31] Klopp F, Nakamura S, Nakano F and Nomura Y 2003 *Ann. Henri Poincaré* **4** 795
[32] Cramér H and Leadbetter M R 1967 *Stationary and Related Stochastic Processes* (New York: Wiley)
[33] Cornfeld I P, Fomin S V and Sinai Ya G 1982 *Ergodic Theory* (New York: Springer)
[34] Iwatsuka A 1985 *Publ. Res. Inst. Math. Sci., Kyoto Univ.* **21** 385
[35] Müller J E 1992 *Phys. Rev. Lett.* **68** 385

[36] Krakovsky A 1996 *Phys. Rev.* B **53** 8469
[37] Măntoiu M and Purice R 1997 *Commun. Math. Phys.* **188** 691
[38] Reijniers J and Peeters F 2000 *J. Phys.: Condens. Matter* **12** 9771
[39] Nogaret A, Bending S J and Henini M 2000 *Phys. Rev. Lett.* **84** 2231
[40] Sim H-S, Chang K J, Kim N and Ihm G 2001 *Phys. Rev.* B **63** 125329
[41] Lawton D, Nogaret A, Makarenko M V, Kibis O V, Bending S J and Henini M 2002 *Physica* E **13** 699
[42] Leschke H, Ruder R and Warzel S 2002 *J. Phys. A: Math. Gen.* **35** 5701
[43] Cycon H, Froese R G, Kirsch W and Simon B 1987 *Schrödinger Operators* (Berlin: Springer)
[44] Fock V 1928 *Z. Phys.* **47** 446
[45] Landau L 1930 *Z. Phys.* **64** 629
[46] Asch J and Knauf A 1998 *Nonlinearity* **11** 175
[47] Leschke H, Warzel S and Weichlein A 2004 *Preprint*
[48] Callaway J 1991 *Quantum Theory of the Solid State* 2nd edn (Boston, MA: Academic) sections 6.1.1 and 6.1.2
[49] Ueki N 2000 *Ann. Henri Poincaré* **1** 473